EXTREMAL QUASICONFORMAL MAPPINGS WITH PRESCRIBED BOUNDARY VALUES

BY
RICHARD S. HAMILTON

1. Let R be a Riemann surface whose universal covering space is conformally equivalent to the unit disk. We can regard R as the interior of a Riemann surface with boundary R^* whose boundary is as large as possible (see §3). A quasiconformal map f of R onto another Riemann surface S has a unique continuous extension f^* mapping R^* onto S^*. Two quasiconformal maps f and g of R onto S are homotopic modulo the boundary if $f^*=g^*$ on R^*-R and there exists a homotopy between f^* and g^* which is constant on R^*-R. If R^*-R is empty then f is just homotopic to g.

Let K be the complex cotangent bundle of R. Then $\beta(f)$, the Beltrami differential of f, is an element of $L^\infty(\bar{K}K^{-1})$, the Banach space of all essentially bounded sections of the bundle $\bar{K}K^{-1}$. (Locally $\nu \in L^\infty(\bar{K}K^{-1})$ is given by $\nu_\alpha\, d\bar{z}_\alpha/dz_\alpha$.) $L^\infty(\bar{K}K^{-1})$ is the dual of $L^1(K^2)$, the Banach space of integrable sections of the bundle K^2. (Locally $\varphi \in L^1(K^2)$ is given by $\varphi_\alpha\, dz_\alpha^2$.) Let $A(R)$ denote the closed subspace of $L^1(K^2)$ of all integrable analytic quadratic differentials on R. By $\|\beta(f) \mid A(R)\|$ we denote the norm of the restriction of $\beta(f)$, regarded as a linear functional on $L^1(K^2)$, to the closed subspace $A(R)$.

The quasiconformal map $f\colon R \to S$ is extremal if $\|\beta(f)\| \leqq \|\beta(g)\|$ for all maps g homotopic to f modulo the boundary. The main result is

THEOREM 1. *If $f\colon R \to S$ is extremal then $\|\beta(f) \mid A(R)\| = \|\beta(f)\|$.*

COROLLARY 1. *If R is a compact surface minus a finite number of points, then $A(R)$ is finite dimensional and consists of all meromorphic quadratic differentials on the compact surface with poles of order at most one at the deleted points. Furthermore there exists a nonzero $\theta \in A(R)$ and a constant c with $\beta(f)=c\bar{\theta}/|\theta|$.*

REMARK. This result is classical and due to Teichmüller.

Proof. Since $A(R)$ is finite dimensional, by Theorem 1 there exists a nonzero $\theta \in A(R)$ with $\int_R \beta(f)\theta = \int_R \|\beta(f)\|\, |\theta|$. This can happen only if $\beta(f)\theta = \|\beta(f)\|\, |\theta|$, and therefore $\beta(f) = \|\beta(f)\| \bar{\theta}/|\theta|$.

COROLLARY 2. *If $f\colon R \to S$ is extremal then $\|\beta(f) \mid R-K\| = \|\beta(f)\|$ for every compact proper subset K of R.*

Proof. By Theorem 1 we can find a sequence $\theta_n \in A(R)$ with $\|\theta_n\|=1$ and $\int_R \beta(f)\theta_n \to \|\beta(f)\|$. Since the value of an analytic function at a point is the average

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of its values over a disk centered at that point, the θ_n are uniformly bounded on every compact subset. Using Cauchy's formula their derivatives are uniformly bounded also, and by passing to a subsequence we may assume that the θ_n converge uniformly on compact subsets to some $\theta \in A(R)$. Suppose $\|\beta(f) \mid R-K\| < \|\beta(f)\|$. Since

$$\left| \int_R \beta(f)\theta_n \right| \leqq \|\beta(f)\| \int_K |\theta_n| + \|\beta(f) \mid R-K\| \int_{R-K} |\theta_n|$$

we must have $\int_{R-K} |\theta_n| \to 0$, and the θ_n converge to θ in L^1 norm. We then have $\|\theta\|=1$ and $\beta(f)(\theta)=\|\beta(f)\|$, and we can repeat the previous argument to show $\beta(f)=\|\beta(f)\|\bar{\theta}/|\theta|$. This gives a contradiction.

REMARK. Given any quasiconformal map $f\colon R \to S$, it follows from the usual compactness properties of quasiconformal mappings that there exists at least one extremal map homotopic to f modulo the boundary. Strebel [7] has shown that when R is the unit disk such an extremal need not have the form $\beta(f)=c\bar{\theta}/|\theta|$ and need not be unique.

2. The proof of Theorem 1 will be modeled upon the following general result.

THEOREM 2. *Let B be a Banach space and B^* its dual. Let M be a C^1 submanifold of B^*. Suppose that the dual norm in B^* assumes its minimum, or maximum, on M at a point x in M, and that there exists a closed subspace A of B such that the tangent space to M at x is the subspace of B^* orthogonal to A. Then $\|x \mid A\|=\|x\|$.*

Proof. Suppose $\|x \mid A\| < \|x\|$. By the Hahn-Banach extension theorem we can find a linear functional y in B^* with $y \mid A = x \mid A$ and $\|y\|=\|x \mid A\| < \|x\|$. Since $y-x$ vanishes on A, we can construct a C^1 path $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0)=x$ and $D\alpha(0)(1)=y-x$.

Then for sufficiently small positive t

$$\|\alpha(t)-\alpha(0)-D\alpha(0)(t)\|/t < \|x\|-\|y\|$$

since $\|x\|-\|y\|>0$. But then

$$\begin{aligned}\|\alpha(t)\| &< \|\alpha(0)+D\alpha(0)(t)\|+(\|x\|-\|y\|)t \\ &\leqq \|x+t(y-x)\|+(\|x\|-\|y\|)t \\ &\leqq (1-t)\|x\|+t\|y\|+t\|x\|-t\|y\| = \|x\|\end{aligned}$$

and the norm does not assume its minimum on M at x. Similarly, for sufficiently small negative t, $\|\alpha(t)\|>\|x\|$ and the norm does not assume its maximum on M at x either.

In the case where B is a Hilbert space this condition yields a familiar result.

COROLLARY 3. *If B is a Hilbert space, then x is orthogonal to the tangent space to M at x.*

Proof. Since $\|x \mid A\|=\|x\|$, $x \in A$.

3. We can represent R as the quotient of the unit disk D under the Fuchsian group Γ. Write the boundary of D as the disjoint union of the closed set $\Lambda(\Gamma)$ of limits of fixed points of Γ and the relatively open set $\Phi(\Gamma)$ of points of discontinuity of Γ. Then $R^* = D \cup \Phi(\Gamma)/\Gamma$ is a Riemann surface with boundary whose interior is R, and any such surface is (conformally equivalent to) an open subset of R^*. Thus the boundary of R^* is as large as possible.

If $f\colon R \to S$ is a quasiconformal map, we may represent S also as the quotient of the disk by another Fuchsian group Δ, and the map f is covered by a quasiconformal map F of the unit disk to itself which has a continuous extension F^* on the closed disk covering a continuous extension $f^*\colon R^* \to S^*$ of f.

THEOREM 3. *Two maps $f, g\colon R \to S$ are homotopic modulo the boundary if and only if they can be covered by maps F^* and G^* of the disk to itself which agree on the boundary of the disk.*

Proof. First suppose f and g are homotopic modulo the boundary, and let $h^*(t)\colon R^* \to S^*$ be the homotopy with $h^*(0) = f^*$, $h^*(1) = g^*$ and h^* constant on $R^* - R$. Then by the Covering Homotopy Theorem we can cover $h^*(t)$ with a homotopy $H^*(t)\colon D \to D$ which is constant on $\Phi(\Gamma)$.

For each $\gamma \in \Gamma$ there exists a $\delta \in \Delta$ with $H^*(0)\gamma = \delta H^*(0)$. Fix x in the interior of D and consider the two curves in D given by $H^*(t)\gamma(x)$ and $\delta H^*(t)x$. Since both are liftings of the same curve in S and both have the same initial point, they must agree. Thus $H^*(t)\gamma = \delta H^*(t)$ for all t.

But then $H^*(t)$ must be constant on the fixed points of Γ, and hence on the whole boundary of the disk. Therefore $F^* = H^*(0)$ and $G^* = H^*(1)$ cover f and g and agree on the boundary of the disk.

Conversely suppose F^* and G^* agree on the boundary of the disk. Define $H^*(t)(z)$ to be the point which divides the noneuclidean line segment between $F^*(z)$ and $G^*(z)$ in the ratio $t : (1-t)$. Then $H^*(t)$ covers a homotopy

$$h^*(t)\colon R^* \to S^*$$

between f^* and g^* constant on $R^* - R$.

Let $f_n\colon R \to S$ be a sequence of maps homotopic to f modulo the boundary, with $\|\beta(f_n)\| \leqq k < 1$. Cover f and f_n with maps F^* and F_n^* of the closed disk to itself which all agree on the boundary, and with $\|\beta(F_n)\| \leqq k < 1$. Some subsequence of the F_n^* will converge uniformly to a quasiconformal map G^* (see Ahlfors [1]) which agrees with F^* on the boundary of the disk, and which covers a quasiconformal map $g\colon R \to S$ homotopic to f modulo the boundary. Moreover $\|\beta(g)\| \leqq \liminf \|\beta(f_n)\|$ so we may choose the f_n to make g extremal.

4. In order to apply Theorem 2, I need the following result, which occurs (implicitly) in Bers [3].

THEOREM 4. *Let N be the set of all Beltrami differentials of quasiconformal maps of R onto itself homotopic to the identity map modulo the boundary. Then N is an analytic submanifold of $L^\infty(\bar{K}K^{-1})$ in a neighborhood of zero whose tangent space at zero is the subspace orthogonal to $A(R)$.*

For completeness I shall outline the proof. Let D be the unit disk and D' its complement in the sphere. Define

$$P_n\mu(z) = (n!/2\pi i)\int_D \mu(\zeta)(\zeta-z)^{-n-1}\,d\zeta \wedge d\bar{\zeta}.$$

For $\mu \in L^\infty(D)$, $P_n\mu$ is analytic in D' with a zero at infinity of order at least $n+1$, and $d/dzP_n\mu = P_{n+1}\mu$ in D'. Moreover for $\mu \in L^p(D)$, any $p>2$, $P_0\mu$ is Hölder-continuous in the entire sphere and has generalized derivatives $\partial/\partial\bar{z}P_0\mu=\mu$ and $\partial/\partial zP_0\mu = P_1\mu$. In D, P_1 is a singular integral and by the Calderón-Zygmund inequality it is a bounded linear operator of $L^p(D)$ into itself, whose norm, by the Riesz convexity theorem, approaches 1 as p approaches 2. We can then prove (see Ahlfors [2, p. 97]) that

$$w(\mu)(z) = z+P_0(I-\mu P_1)^{-1}\mu(z)$$

is a quasiconformal map of the sphere to itself with Beltrami differential μ on D and analytic on D'.

Remembering that $R=D/\Gamma$, let $L^\infty(D,\Gamma)=L^\infty(\bar{K}K^{-1})$ be the Banach subspace of $\mu \in L^\infty(D)$ with $\mu=(\mu\circ\gamma)\arg^{-2}\gamma'$ for all $\gamma \in \Gamma$. Also let $B(D',\Gamma)$ be the Banach space of all analytic functions φ in D' with a zero of order at least 4 at infinity, which satisfy $\varphi=(\varphi\circ\gamma)(\gamma')^2$ and whose norm $\sup(z\bar{z}-1)^2|\varphi(z)|$ is finite. This is just the norm of the quadratic differential $\varphi(z)\,dz^2$ in the Poincaré metric on D'. Let $[f]$ denote the Schwarzian derivative of f. Since μ is invariant under Γ, for each $\gamma \in \Gamma$ the map $w(\mu)\circ\gamma\circ w(\mu)^{-1}$ is an analytic homeomorphism of the sphere and hence is itself a Möbius transformation δ. Then $w(\mu)\circ\gamma=\delta\circ w(\mu)$, and

$$([w(\mu)]\circ\gamma)(\gamma')^2 = [w(\mu)].$$

Moreover by a theorem of Nehari [6] on schlicht mappings $\sup(z\bar{z}-1)^2|[w(\mu)]| \leqq 6$. Hence $\Lambda(\mu)=[w(\mu)]$ belongs to $B(D',\Gamma)$ and $\|\Lambda(\mu)\| \leqq 6$.

LEMMA 1. *$\Lambda: L^\infty(D,\Gamma) \to B(D',\Gamma)$ is a complex analytic map and $D\Lambda(0)=P_3$.*

Proof. Fix a point $z \in D$. Since $(I-\mu P_1)^{-1}\mu$ is a uniformly convergent power series in μ, $w(\mu)(z)$, $d/dzw(\mu)(z),\ldots, d^3/dz^3w(\mu)(z)$ are all analytic functions of μ. Therefore so is $\Lambda(\mu)(z)$. Let γ be the circle of radius 1 in the t-plane. By the Cauchy integral formula, for $\|\nu\| < 1-\|\mu\|$,

$$D\Lambda(\mu)(\nu)(z) = (1/2\pi i)\int_\gamma \Lambda(\mu+t\nu)(z)/t^2\,dt.$$

Since $|\Lambda(\mu+t\nu)(z)| \leqq 6(z\bar{z}-1)^{-2}$, $D\Lambda(\mu)$ is a bounded complex linear map of $L^\infty(D, \Gamma)$ into $B(D', \Gamma)$. Also for $|c| \leqq 1/2$

$$\begin{aligned}|\Lambda(\mu+c\nu)(z)-\Lambda(\mu)(z)-D\Lambda(\mu)(c\nu)(z)| \\ &\leqq (1/2\pi)\int_\gamma |\Lambda(\mu+t\nu)(z)|\ |(t-c)^{-1}-t^{-1}-ct^{-2}|\ dt \\ &\leqq 12c^2(z\bar{z}-1)^{-2}.\end{aligned}$$

Therefore $\|\Lambda(\mu+c\nu)-\Lambda(\mu)-D\Lambda(\mu)(c\nu)\| \leqq 12c^2$ so Λ is in fact differentiable with derivative $D\Lambda(\mu)$. Since $D\Lambda(\mu)$ is complex-linear, Λ is analytic. By evaluating at z again we may compute $Dw(0)(\mu)(z) = P_0\mu(z)$ and $D\Lambda(0)(\mu)(z) = P_3\mu(z)$. Therefore $D\Lambda(0) = P_3$.

Define a continuous linear map $S\colon B(D', \Gamma) \to L^\infty(D, \Gamma)$ by

$$S\varphi(z) = c(1-z\bar{z})^2\bar{z}^{-4}\varphi(\bar{z}^{-1}).$$

Using a reproducing formula for analytic functions (see Bers [3, Lecture 3, p. 6]), $D\Lambda(0)\circ S$ is the identity for an appropriate choice of the constant c. This proves that $D\Lambda(0)$ maps $L^\infty(D, \Gamma)$ onto $B(D', \Gamma)$ and its kernel is a closed split subspace. By the inverse function theorem $\Lambda^{-1}(0)$ is an analytic submanifold in a neighborhood of zero.

LEMMA 2. $\Lambda^{-1}(0) = N$.

Proof. First suppose $\mu \in \Lambda^{-1}(0)$. Then the Schwarzian derivative of $w(\mu)$ is zero on D', so $w(\mu)$ agrees on D' with a Möbius transformation A. Let $w = A^{-1}\circ w(\mu)$. Then w is μ-quasiconformal on D and the identity on D'. Since $\mu \in L^\infty(D, G)$, w covers a μ-quasiconformal map of R onto itself homotopic to the identity modulo the boundary.

Conversely any such map can be lifted to a quasiconformal map w of D onto itself which leaves the boundary fixed. Let $\mu = \beta(w)$, and extend w to be the identity in D'. Then $w(\mu)w^{-1}$ is an analytic one-to-one map of the sphere onto itself and therefore is a Möbius transformation whose Schwarzian derivative in D' is $\Lambda(\mu) = 0$.

Finally

$$D\Lambda(0)(\mu) = P_3\mu = (6/2\pi i)\int_D \mu(\zeta)/(\zeta-z)^4\, d\zeta \wedge d\bar{\zeta}$$

and the functions $(\zeta-z)^{-4}$ are dense in $A(D)$, the integrable analytic functions in D. Moreover we know (see Earle [4]) that each element of $A(R)$ is a Poincaré series of an element of $A(D)$. Therefore $T_0N = \text{Ker } D\Lambda(0)$ is the subspace of $L^\infty(\bar{K}K^{-1})$ orthogonal to $A(R)$.

5. The composition of two quasiconformal maps is again quasiconformal. If $\mu=\beta(f)$ and $\nu=\beta(g)$ then

$$\begin{aligned}\partial(g\circ f)/\partial z &= \partial g/\partial w\ \partial f/\partial z+\partial g/\partial \bar{w}\ \partial \bar{f}/\partial z\\ &= \partial g/\partial w\ \partial f/\partial z(1+\bar{\mu}\nu\arg^{-2}\partial f/\partial z)\\ \partial(g\circ f)/\partial \bar{z} &= \partial g/\partial w\ \partial f/\partial \bar{z}+\partial g/\partial \bar{w}\ \partial \bar{f}/\partial \bar{z}\\ &= \partial g/\partial w\ \partial f/\partial z(\mu+\nu\arg^{-2}\partial f/\partial z).\end{aligned}$$

Here $f^{\#}\nu=\nu\arg^{-2}\partial f/\partial z$ is the pull-back of $\nu\, d\bar{w}/dw$ as a tensor. We then have

$$\beta(g\circ f) = (\mu+f^{\#}\nu)/(1+\bar{\mu}f^{\#}\nu).$$

We may regard this as a calculation in local coordinates for Riemann surfaces. Let R, S, and T be Riemann surfaces, $f\colon R\to S$ and $g\colon S\to T$ quasiconformal maps, and K and J the complex cotangent bundles on R and S. The tensor pull-back defines a linear isometric isomorphism $f^{\#}\colon L^{\infty}(\bar{J}J^{-1})\to L^{\infty}(\bar{K}K^{-1})$. It is an isometry because $|f^{\#}\nu_\alpha|=|\nu_\beta|\circ f_{\beta\alpha}$, and an isomorphism because $(f^{\#})^{-1}=(f^{-1})^{\#}$. Then if $\mu=\beta(f)$ and $\nu=\beta(g)$ we have as before

$$\beta(g\circ f) = (\mu+f^{\#}\nu)/(1+\bar{\mu}f^{\#}\nu).$$

This suggests that we define a map of the Beltrami differentials on S to the Beltrami differentials on R by

$$C(f)(\nu) = (\mu+f^{\#}\nu)/(1+\bar{\mu}f^{\#}\nu).$$

The Beltrami differentials on R are the points in the unit ball in $L^{\infty}(\bar{K}K^{-1})$, which I denote by $B(R)$. The map $C(f)\colon B(S)\to B(R)$ is analytic. Indeed $(1+\bar{\mu}f^{\#}\nu)^{-1}$ admits a uniformly convergent power series since $\|\bar{\mu}f^{\#}\nu\|<1$. Moreover $\beta(g\circ f)=C(f)\beta(g)$, and since every Beltrami differential is the Beltrami differential of some quasiconformal map, it follows that $C(g\circ f)=C(g)\circ C(f)$. Therefore, $C(f)^{-1}=C(f^{-1})$ and $C(f)$ is bi-analytic.

Let d be the Poincaré metric in the disk D, given by

$$d(z,w) = (1/2)\log(1+r)/(1-r)\quad\text{where } r = |(z-w)/(1-\bar{z}w)|.$$

If ν and π are two Beltrami differentials and α and β are two coordinate charts then since

$$\nu_\beta = \nu_\alpha\arg^2(dz_\beta/dz_\alpha)\quad\text{and}\quad \pi_\beta = \pi_\alpha\arg^2(dz_\beta/dz_\alpha)$$

the number $d(\nu(x),\pi(x))=d(\nu_\alpha(x),\pi_\alpha(x))$ is invariantly defined for almost all x. Define a metric on $B(R)$, the Beltrami differentials on R, by

$$\tau(\nu,\pi) = \operatorname{ess\,sup} d(\nu(x),\pi(x))$$

where the essential supremum is taken over almost all x in R. This metric is natural in the sense that it makes each map $C(f)\colon B(S)\to B(R)$ an isometry (if we define a metric τ on $B(S)$ in the same way). To establish this result it is necessary only to observe that in terms of a local coordinate chart the map $C(f)$ is induced by a Möbius transformation from the unit disk in each fibre of $L^{\infty}(\bar{J}J^{-1})$ to the unit disk in the corresponding fibre of $L^\infty(\bar{K}K^{-1})$, and the Poincaré metric is invariant under Möbius transformations. This metric is shown to be induced by a Finsler structure on $B(R)$ by Earle and Eells in [5].

We shall need the following estimate comparing the metric τ with the L^∞ norm.

THEOREM 5. *Let ν and π be Beltrami differentials. Then*

$$\tau(\nu, \pi) - \tau(\nu, 0) \leqq \|\nu - \pi\| - \|\nu\| + o(\|\pi\|)$$

where $o(t)/t \to 0$ as $t \to 0$.

Proof. The Poincaré metric is very close to the Euclidean metric at the origin, so that

$$d(z, w) - d(z, 0) \leqq |z - w| - |z| + o(|w|)$$

where $o(t)/t \to 0$ as $t \to 0$. To prove this, regard $d(z, w)$ and $|z-w|$ as two functions of w for fixed $z \neq 0$ and evaluate the derivatives $\partial/\partial w$ at $w=0$. In both cases a laborious calculation yields $-\bar{z}/2|z|$. Since $d(z, w)$ and $|z-w|$ are real the derivatives $\partial/\partial \bar{w}$ are obtained by conjugation. Hence both partial derivatives agree at $w=0$ and are continuous, so the estimate follows from the Mean Value Theorem. On the other hand, if $z=0$ we may regard

$$d(0, w) = (1/2) \log (1+|w|)/(1-|w|)$$

as a function of $|w|$, and taking its derivative at $|w|=0$ we again obtain the required estimate.

By replacing $o(t)$ by $\sup \{o(u) \mid 0 \leqq u \leqq t\}$ we may assume that the error estimate $o(t)$ is monotone nondecreasing. Then for the metric τ we have

$$\begin{aligned} \tau(\nu, \pi) &= \text{ess sup } d(\nu(x), \pi(x)) \\ &\leqq \text{ess sup } \{d(\nu(x), 0) + |\nu(x) - \pi(x)| - |\nu(x)| + o(|\pi(x)|)\} \\ &\leqq \text{ess sup } \{d(\nu(x), 0) - |\nu(x)|\} + \text{ess sup } |\nu(x) - \pi(x)| + \text{ess sup } o(|\pi(x)|). \end{aligned}$$

But $d(z, 0) - |z|$ is a monotonic increasing function of $|z|$, since

$$d/dr (1/2) \log (1+r)/(1-r) = 1/(1-r^2).$$

Therefore

$$\tau(\nu, \pi) \leqq \tau(\nu, 0) - \|\nu\| + \|\nu - \pi\| + o(\|\pi\|)$$

which proves the theorem.

6. Let N be as before the set of all Beltrami differentials of quasiconformal maps of R onto itself homotopic to the identity modulo the boundary.

THEOREM 6. *Let f: $R \to S$ be extremal and $\mu = \beta(f)$. Then $\tau(\mu, 0) \leqq \tau(\mu, \pi)$ for all $\pi \in N$.*

Proof. Suppose $\pi \in N$ and $\tau(\mu, \pi) < \tau(\mu, 0)$. We know that $\pi = \beta(g)$ for some quasiconformal map g: $R \to R$ homotopic to the identity modulo the boundary.

If $H: R\times[0,1]\to R$ is a homotopy between g and the identity fixing the boundary then $f\circ g^{-1}\circ H$ is a homotopy between f and the map $k=f\circ g^{-1}$ which leaves the boundary fixed. Since $k\circ g=f$, $C(g)\beta(k)=\beta(f)$. Let $\mu=\beta(f)$ and $\lambda=\beta(k)$. Since $C(g)$ is an isometry and $C(g)0=\beta(g)=\pi$,

$$\tau(\lambda,0) = \tau(C(g)\lambda, C(g)0) = \tau(\mu,\pi) < \tau(\mu,0).$$

But $\tau(\mu,0)$ is a monotone increasing function of $\|\mu\|$ since $d(z,0)$ is a monotone increasing function of z. Therefore $\|\lambda\|<\|\mu\|$. Since $\lambda=\beta(k)$ and k is homotopic to f modulo the boundary, f is not extremal.

It is now easy to complete the proof of Theorem 1 by imitating the proof of Theorem 2, using the estimate in Theorem 5. Suppose that $f: R\to S$ is quasiconformal with Beltrami differential $\mu=\beta(f)$ but that $\|\mu \mid A(R)\|<\|\mu\|$. By the Hahn-Banach Theorem we can find $\nu\in L^\infty(\bar{K}K^{-1})$ with $\nu \mid A(R)=\mu \mid A(R)$ and $\|\nu\|=\|\mu \mid A(R)\|<\|\mu\|$. Then $\mu-\nu\in A(R)^\perp$ and $A(R)^\perp$ is the tangent space at 0 to the analytic submanifold N. Consequently we can find a C^1 path $\alpha: (-\varepsilon,\varepsilon)\to N$ with $\alpha(0)=0$ and $D\alpha(0)(1)=\mu-\nu$. Then, restricting our attention to positive t,

$$\|\alpha(t)-\alpha(0)-D\alpha(0)(t)\|/t = \|\alpha(t)-t\mu+t\nu\|/t\to 0 \quad \text{as } t\to 0.$$

This makes $\|\alpha(t)\|\leqq Kt$ for some constant K and all sufficiently small t, so $o(\|\alpha(t)\|)/t\to 0$ as $t\to 0$ as well.

Now by the estimate of Theorem 5

$$\tau(\mu,\alpha(t))-\tau(\mu,0) \leqq \|\mu-\alpha(t)\|-\|\mu\|+o(\|\alpha(t)\|).$$

Also $\|\mu-\alpha(t)\|\leqq(1-t)\|\mu\|+t\|\nu\|+\|\alpha(t)-t\mu+t\nu\|$. Combining these inequalities with the estimates above we see that

$$\tau(\mu,\alpha(t))-\tau(\mu,0) \leqq \beta(t)-t(\|\mu\|-\|\nu\|)$$

where $\beta(t)/t\to 0$ as $t\to 0$. Since $\|\mu\|-\|\nu\|>0$ we must have $\tau(\mu,\alpha(t))<\tau(\mu,0)$ for all sufficiently small pcsitive t. Then since $\alpha(t)\in N$ it follows from Theorem 6 that the map $f: R\to S$ is not extremal.

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Cornell University,
Ithaca, New York